Kinetic Equations

Solution to the Exercises

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Exercise 1

Let f be a measurable function such that $d_f(\gamma) < +\infty$ for any $\gamma > 0$. Let $M > 0$. Define $f_M^-(x) := \chi_{\{|f|\leqslant M\}}(x) f(x)$ and $f_M^+ := f - f_M^-$. Prove that

$$d_{f_M^-}(\gamma) := \begin{cases} d_f(\gamma) - d_f(M), & \text{if } \gamma < M, \\ 0, & \text{if } \gamma \geqslant M, \end{cases} \tag{1}$$

$$d_{f_M^+}(\gamma) := \begin{cases} d_f(M), & \text{if } \gamma \leqslant M, \\ d_f(\gamma), & \text{if } \gamma > M. \end{cases} \tag{2}$$

Proof. Recall that $d_f(\gamma) = |\{x| \ |f(x)| > \gamma\}|$

Let first $\gamma > M$; then if $\left|f_M^+(x)\right| > \gamma$ we have that $f_M^+(x) = f(x)$ and therefore

$$d_{f_M^+}(\gamma) = \left|\left\{x| \ \left|f_M^+(x)\right| > \gamma\right\}\right| = |\{x| \ |f(x)| > \gamma\}| = d_f(\gamma). \tag{3}$$

Let then $\gamma \leqslant M$. If x is such that $\left|f_M^+(x)\right| > \gamma$, this implies that $f_M^+(x) \neq 0$ $(\gamma > 0)$. Therefore, $f_M^+(x) = f(x)$ and $|f(x)| > M$, meaning

$$d_{f_M^+}(\gamma) = \left|\left\{x| \ \left|f_M^+(x)\right| > \gamma\right\}\right| = |\{x| \ |f(x)| > M\}| = d_f(M). \tag{4}$$

We consider now f_M^-. Let $\gamma \geqslant M$. If x is such that $\left|f_M^-(x)\right| > \gamma$, then we have $f_M^-(x) \neq 0$ and therefore $f_M^-(x) = f(x)$ and $|f(x)| \leqslant M$. But this gives a contradiction, because $|f(x)| \leqslant M \leqslant \gamma < |f(x)|$, which is absurd. Therefore there is not such an x and $d_{f_M^-}(\gamma) = 0$.

Let finally $\gamma < M$. Then

$$\begin{aligned} \left|f_M^-(x)\right| > \gamma, \ \gamma < M &\iff |f(x)| > \gamma, \ |f(x)| \leqslant M, \gamma < M & (5) \\ &\iff \gamma < |f(x)| \leqslant M, & (6) \end{aligned}$$

and therefore

$$\begin{aligned} d_{f_M^-}(\gamma) &= \left|\left\{x| \ \left|f_M^+(x)\right| > \gamma\right\}\right| = |\{x| \ |f(x)| > M\} \setminus \{x| \ |f(x)| > \gamma\}| & (7) \\ &= d_f(M) - d_f(\gamma). & (8) \end{aligned}$$

□

Exercise 2

Prove the weak Young inequality for $p = 1$, i.e., that for any $q \in (1, +\infty)$ there exists a constant $C_q > 0$ such that for any $f \in L^1\left(\mathbb{R}^d\right)$, $g \in L^{q,\infty}\left(\mathbb{R}^d\right)$

$$\|f * g\|_{L^{q,\infty}(\mathbb{R}^d)} \leqslant C_q \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^{q,\infty}(\mathbb{R}^d)} . \tag{9}$$

Proof. Similar as we did in the last exercise session, we first notice that, with g_M^+ and g_M^- defined as above, we get that $\|g_M^-\|_{L^\infty(\mathbb{R}^d)} \leqslant M$ and therefore

$$\left|f * g_M^-(x)\right| \leqslant \|f\|_{L^1(\mathbb{R}^d)} \|g_M^-\|_{L^\infty(\mathbb{R}^d)} \leqslant M \|f\|_{L^1(\mathbb{R}^d)} . \tag{10}$$

We then fix $M := \frac{\gamma}{2\|f\|_{L^1(\mathbb{R}^d)}}$; as a consequence we get $d_{f*g_M^-}\left(\frac{\gamma}{2}\right) = 0$.

Recall that we saw in class that

$$\|g_M^+\|_{L^1(\mathbb{R}^d)} \leqslant \frac{q}{q-1} M^{1-q} \|g\|^q_{L^{q,\infty}(\mathbb{R}^d)} , \tag{11}$$

and as a consequence we get

$$\|f * g_M^+\|_{L^1(\mathbb{R}^d)} \leqslant \|f\|_{L^1(\mathbb{R}^d)} \|g_M^+\|_{L^1(\mathbb{R}^d)} \leqslant \frac{q}{q-1} M^{1-q} \|f\|_{L^1(\mathbb{R}^d)} \|g\|^q_{L^{q,\infty}(\mathbb{R}^d)} . \tag{12}$$

We therefore get

$$d_{f*g}(\gamma) \leqslant d_{f*g_M^+}\left(\frac{\gamma}{2}\right) + d_{f*g_M^-}\left(\frac{\gamma}{2}\right) = d_{f*g_M^+}\left(\frac{\gamma}{2}\right) \tag{13}$$

$$\leqslant \frac{2\left\|f * g_M^+\right\|_{L^1(\mathbb{R}^d)}}{\gamma} \leqslant \frac{2q}{\gamma(q-1)} M^{1-q} \|f\|_{L^1(\mathbb{R}^d)} \|g\|^q_{L^{q,\infty}(\mathbb{R}^d)} \tag{14}$$

$$= \left(\frac{\gamma}{2}\right)^{-q} \|f\|^q_{L^1(\mathbb{R}^d)} \|g\|^q_{L^{q,\infty}(\mathbb{R}^d)} , \tag{15}$$

Which implies

$$\|f * g\|_{L^{1,\infty}(\mathbb{R}^d)} \leqslant 2 \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^{q,\infty}(\mathbb{R}^d)} . \tag{16}$$

□

Exercise 3

Let $[a, b] \subset \mathbb{R}$ be a compact interval. Assume that $f \in C^1([a, b])$ such that $f(x) > 0$ for any $x \in [a, b]$. Suppose that there exists $C > 0$ such that $f' \in C([a, b])$ satisfies

$$f'(x) \leqslant C f(x) \left[1 + |\log(f(x))|\right], \qquad \forall x \in [a, b] . \tag{17}$$

Prove that this implies that

$$f(x) \leqslant \exp\left((1 + |f(a)|) \exp\left(C(x - a)\right)\right), \qquad \forall x \in [a, b] . \tag{18}$$

Proof. First of all, notice that there exist $M > 0$ such that $[a, b]$ can be written as

$$[a, b] = \{a\} \cup \bigcup_{j=1}^{M-1} (a_j, a_{j+1}] \tag{19}$$

with $a_1 = a$, $a_N = b$, $f(a_j) = 1$ for all j and $|f(x) - 1| > 0$.

In particular, $f(x) - 1$ cannot change sign in (a_j, a_{j+1}) for any j.

Consider now j fixed. If $f(x) - 1 \leqslant 0$, we get $f(x) \leqslant 1$. Given that $x - a \geqslant 0$ and therefore $e^{(1+|f(a)|)e^{C(x-a)}} \geqslant 1$, this implies the result in this case. On the other hand, if $f(x) - 1 \geqslant 0$, this means $|\log(f(x))| = \log(f(x))$. This implies that (17) can be rewritten as

$$\partial_x [\log(1 + \log(f(x)))] \leqslant C. \tag{20}$$

We can then integrate it explicitly, and recalling that $f(a_j) = 1$ we get

$$\log(1 + \log(f(x))) \leqslant \log(1 + \log(f(a_j))) + \int_{a_j}^{x} \partial_x [\log(1 + \log(f(y)))] \, dy \tag{21}$$

$$\leqslant \log(1 + \log(f(a_j))) + C(x_j - a_j). \tag{22}$$

Now, the first term is either zero or bounded by $\log(1 + |\log(f(a))|) \geqslant 0$, and given that $a_j \geqslant a$ we get

$$\log(1 + \log(f(x))) \leqslant \log(1 + |\log(f(a))|) + C(x - a). \tag{23}$$

By definition of log this implies

$$f(x) \leqslant e^{(1+|f(a)|)e^{C(x-a)}}. \tag{24}$$

□